Nested Saturation with Guaranteed Real Poles[1]

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Abstract

The global stabilization of asymptotically null-controllable linear systems with bounded controls has been studied extensively. An early contribution was by Teel [6] who proposed a set of nested saturators to globally asymptotically stabilize the special case of n-integrators with one input. Using this law however, the closed loop system pole locations depend on the choice of coordinate transformation used to arrive at the control law. In this paper we suggest an approach that allows the designer to pick transformations that facilitate the placement of the closed loop poles on the negative real axis.

1 Introduction

The problem addressed involves the global stabilization of a chain of integrators

$$\dot{x}_1 = x_2, \cdots, \dot{x}_n = u \tag{1}$$

The system given by (1) is a subset of a class of systems that are said to be *asymptotically null-controllable with bounded controls* [3, 1]. This property was shown in [2] to be equivalent to the system being stabilizable and having all open-loop poles in the closed left-half plane.

It was shown in [4] that it is not possible to globally stabilize integrator chains of order $n > 2$ using a bounded *linear* feedback law. However, it was shown by Teel in [6] that a nonlinear law consisting of nested saturators can guarantee global asymptotic stability for integrator chains of any order n. This control law may be expressed as

$$u = -\sigma_n(h_n(x) + \sigma_{n-1}(h_{n-1}(x) + \cdots + \sigma_1(h_1(x))))$$

where h_i are linear combinations of the state (feedback) and the saturation functions σ_i satisfy certain properties. The existence of such a globally stabilizing control law was established in [6] by choosing *one* set of h_i's such that global asymptotic stability could be proven. The choice of h_i is a design degree of freedom and may be exercised to prescribe pole locations and the linear dynamics when different elements of the control law are saturated.

We observed that the h_i chosen by Teel with *conventional saturation* functions (see Definition 2) results in all the poles of the closed loop system residing at -1 when none of the saturation elements in the control law are saturated. If the k^{th} saturator is the outermost element to be saturated, then the resulting closed loop system has poles at -1 with multiplicity $n-k$ and poles at 0 with multiplicity k, at least until the element comes out of saturation. A discussion on the prescription of performance by pole placement (both real and complex) is provided in [5], however no explicit transformation is provided. Another aspect is the behavior of these poles as different elements of the control law saturate. Ideally, these poles should not change when saturation occurs. Both these properties (pole placement and movement when saturated) are useful if the nested saturation control law is to be employed in practice.

We believe that the simple and elegant nested saturation law can benefit greatly from these properties. Hence, the effort here is to develop a transformation, i.e., a way to select h_i such that closed loop poles for the unsaturated system may be prescribed as $\{-a_1, -a_2, \cdots - a_n\}$, where $a_i \in \mathbb{R} \setminus 0$ and $a_i > 0$ for stability. Additionally, it will be shown that when the outermost saturated element is σ_k, the poles of resulting linear system reside at $\{-a_1, -a_2, \cdots - a_{n-k}, 0_1, 0_2, \cdots 0_k\}$.

2 Main Result

Definition 1 (Linear saturation) *Define constants $(L, M) \in \mathbb{R}+$ such that $0 < L \leq M$. Now, define a function $\sigma : \mathbb{R} \to \mathbb{R}$. σ is said to be a* linear saturation *if it is continuous, nondecreasing and satisfies*

- *a.* $s\sigma(s) > 0 \quad \forall s \neq 0$
- *b.* $\sigma(s) = s \quad$ *when* $|s| \leq L$
- *c.* $|\sigma(s)| \leq M \quad \forall s \in \mathbb{R}$

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Definition 2 (Conventional saturation)
$\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is said to be a conventional saturation *if it has a limit $M \in R+$ such that*

a. $s\sigma(s) > 0 \quad \forall s \neq 0$

b. $\sigma(s) = s \quad$ *when* $|s| \leq M$

c. $|\sigma(s)| = M \quad$ *when* $|s| > M$

Remark 1 *σ is said to be* saturated *when its argument is not in its linear region. For* linear saturation *this occurs when $|s| > L$. For* conventional saturation *this occurs when $|s| > M$.*

Remark 2 Conventional saturation *is a special case of* linear saturation *with $L = M$ and a constant saturation value M.*

Lemma 1 *Consider a chain of n-integrators, given by (1), which may be represented as $\dot{x} = A_x x + B_x u$, with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and*

$$A_x = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad B_x = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \tag{2}$$

then there exists a linear transformation $y = T_{yx}x$ which transforms (1) into $\dot{y} = A_y y + B_y u$ where,

$$A_y = \begin{bmatrix} 0 & a_n & \cdots & \cdots & a_n \\ 0 & 0 & a_{n-1} & \cdots & a_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & a_2 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad B_y = \begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ \vdots \\ a_1 \end{bmatrix} \tag{3}$$

and the elements $a_i \in \mathbb{R} \setminus 0$ with $i = 1 \ldots n$.

Proof: Given a set of coefficients

$$A = \{a_1, a_2, \cdots, a_n\} \tag{4}$$

Let $A_l \subseteq A$ represent a subset containing the first l elements of A. Define a function $F_k^m(A_l)$ which acts over the set A_l. F_k^m is used to generate the product of combinations of elements taken m at a time from A_l. The number of such combinations is given by the binomial coefficient $\binom{l}{m}$. Hence, $F_k^m(A_l)$ may be treated as a *generating function* that outputs the k^{th} combination of the product of m elements taken from the set A_l without repetition and disregarding order. Note that $F_k^0 = 1$.

In order to generate the transformation T_{yx}, define the function $C(l,m)$, with $l \in [0, \cdots, n]$, $m \in [0, \cdots, l]$ and $m \leq l$, over the set of coefficients A given by (4).

$$C(l,m) = \sum_{k=1}^{\bar{C}_m^l} F_k^m(A_l) \tag{5}$$

$$C(l,0) = 1 \tag{6}$$

$\bar{C}_m^l$ is the binomial coefficient $\binom{l}{m}$. The new coordinate system is characterized by

$$y_{n-i} = a_{i+1} \sum_{j=0}^{i} C(i,j) x_{n-j}, \qquad i \in [0, \cdots, n-1] \tag{7}$$

and the transformation T_{yx} is explicitly given by

$$\begin{aligned} T_{yx_{(n-i)(n-j)}} &= a_{i+1} C(i,j) \qquad & i \geq j \\ T_{yx_{(n-i)(n-j)}} &= 0 & i < j \end{aligned} \tag{8}$$

for $i, j \in [0, \cdots, n-1]$. Additionally, T_{yx} is an upper diagonal matrix with non-zero diagonal entries. Hence, $T_{xy} = T_{yx}^{-1}$ exists. Finally, observing that

$$\begin{aligned} \dot{x} &= T_{yx} A_x T_{yx}^{-1} y + T_{yx} B_x u \\ &= A_y y + B_y u \end{aligned}$$

it is enough to verify that $A_y T_{yx} = T_{yx} A_x$ and that $T_{yx} B_x = B_y$. This may be carried out using Equations 2, 3 and 8. ■

Theorem 1 *For the system given by (1). Given any set of positive constants $\{(L_i, M_i)\}$, where $L_i \leq M_i$ for $i = 1, \cdots, n$ and $M_i < \frac{1}{2} L_{i+1}$ for $i = 1, \cdots, n-1$, and for any set of functions $\{\sigma_i\}$ that are* linear saturations *for $\{(L_i, M_i)\}$, there exists a linear coordinate transformation $y = T_{yx}x$ such that the bounded control*

$$u = -\sigma_n(y_n + \sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1))) \tag{9}$$

results in a globally asymptotically stable system.

Proof: In short, use the transformation given by Lemma 1 in the proof of Theorem 2.1 in [6]. It is however restated here for completeness.

Use the coordinate transformation $y = T_{yx}x$ given by Lemma 1 and choose the set of coefficients $a_i > 0$. Substituting the nested saturation law given by Eq. (9) into Eq. (1) and expanding yields the closed loop system

$$\begin{aligned} \dot{y}_1 &= a_n \quad [y_2 + \cdots + y_n - \sigma_n(y_n + \sigma_{n-1}(\cdots \sigma_1(y_1)))] \\ \dot{y}_2 &= a_{n-1}[y_3 + \cdots + y_n - \sigma_n(y_n + \sigma_{n-1}(\cdots \sigma_1(y_1)))] \\ &\vdots \\ \dot{y}_{n-1} &= a_2 \quad [y_n - \sigma_n(y_n + \sigma_{n-1}(\cdots \sigma_1(y_1)))] \\ \dot{y}_n &= - a_1 \sigma_n(y_n + \sigma_{n-1}(\cdots \sigma_1(y_1))) \end{aligned} \tag{10}$$

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The trajectory of y_n is examined first. Choosing a Lyapunov function $V_n = y_n^2$, with $y_n \in \mathbb{R}$. Its derivative $\dot{V}_n$ may be written as

$$\dot{V}_n = -2a_1 y_n \left[\sigma_n(y_n + \sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1)))\right]$$

Noting that $a_i > 0$. Definition 1, conditions (a), (b), imply that y_n and $\sigma_n(\cdot)$ are the same sign only if $y_n + \sigma_{n-1}(\cdot)$ is the same sign as y_n. Condition (c) of Definition 1 applied to σ_{n-1} and having chosen $M_{n-1} < \frac{1}{2}L_n$, it can be seen that $\dot{V}_n < 0$ for all $y_n \notin Q_n = \{y_n : |y_n| \leq \frac{1}{2}L_n\}$. If starting outside Q_n, the trajectory of y_n eventually enters Q_n in *finite time*. Since the RHS of Eq. (10) is globally Lipschitz, the derivatives are bounded resulting in the remaining states $y_1 \cdots y_{n-1}$ remaining bounded for any given *finite time*.

Once y_n has entered Q_n, condition (b) of Definition 1 implies σ_n operates in its linear region because the argument to σ_n is bounded as

$$|y_n + \sigma_{n-1}(\cdot)| \leq \frac{1}{2}L_n + M_{n-1} \leq L_n$$

The equation for the evolution of y_{n-1} is now given by

$$\begin{aligned}\dot{y}_{n-1} &= \cancelto{0}{a_2 y_n - a_2 y_n} - a_2\sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1)) \\ &= -a_2\sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1))\end{aligned}$$

which is similar to the expression for $\dot{y}_n$. Using similar arguments as that used for the evolution of y_n, it can be shown that y_{n-1} enters a set Q_{n-1} in finite time and remains in Q_{n-1} thereafter with all remaining states being bounded. Continuing in the same fashion, it can be shown that every state y_i for $i \in [1, \cdots, n]$, enters a set $Q_i = \{y_i : |y_i| \leq \frac{1}{2}L_i\}$ in finite time and all saturation functions σ_i are operating in their linear regions. Hence after a certain finite amount of time the governing equations, Eq. (10), becomes

$$\begin{aligned}\dot{y}_1 &= -a_n y_1 \\ \dot{y}_2 &= -a_{n-1}(y_1 + y_2) \\ &\vdots \\ \dot{y}_n &= -a_1(y_1 + y_2 + \cdots + y_n)\end{aligned}$$

which is exponentially stable. ∎

Corollary 1 (Pole location) *If the saturators used are* Conventional saturation, *and none of the σ_i are saturated, the poles of the linearized closed loop system reside at $\{-a_1, -a_2, \cdots, -a_n\}$. During periods when the outermost saturated element is the k^{th} saturator, σ_k, the poles of the resulting closed loop linear system reside at $\{-a_1, -a_2, \cdots, -a_{n-k}, 0_1, 0_2, \cdots, 0_k\}$.*

Proof: Using the nested saturation law, the closed-loop n-integrator system may be expressed as

$$\dot{x}_n + \sigma_n(y_n + \sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1))) = 0$$

When the k^{th} saturator is saturated, and $\sigma_{k+1} \cdots \sigma_n$ are not saturated, the closed loop system is given by

$$\dot{x}_n + y_n + y_{n-1} + \cdots y_{k+1} \pm M_k = 0$$

This represents a forced linear system where the forcing function is the constant M_k. Examining the homogeneous part

$$0 = \dot{x}_n + y_n + y_{n-1} + \cdots + y_{k+1}$$

Using Eq. (7) to expand y_i

$$\begin{aligned}0 = \dot{x}_n &+ a_1 \sum_{j=0}^{0} C(0,j)x_{n-j} \\ &+ a_2 \sum_{j=0}^{1} C(1,j)x_{n-j} + \cdots \\ &+ a_{n-k} \sum_{j=0}^{n-(k+1)} C\left(n-(k+1), j\right) x_{n-j}\end{aligned}$$

Noting that $x = x_1, \dot{x} = x_2, \cdots, x^{(n-1)} = x_n, x^{(n)} = \dot{x}_n$, and substituting $p = n - k$ for clarity the characteristic equation may be written as

$$\begin{aligned}\Upsilon(\lambda) = \lambda^n & \\ &+ a_1 C(0,0)\lambda^{n-1} \\ &+ a_2 C(1,0)\lambda^{n-1} + a_2 C(1,1)\lambda^{n-2} \\ &\vdots \\ &+ a_p C(p-1,0)\lambda^{n-1} + \cdots + a_p C(p-1,p-1)\lambda^k\end{aligned}$$

Factoring out λ^k

$$\begin{aligned}\Upsilon(\lambda) = &\lambda^k[\lambda^p \\ &+ a_1 C(0,0)\lambda^{p-1} \\ &+ a_2 C(1,0)\lambda^{p-1} + a_2 C(1,1)\lambda^{p-2} \\ &+ a_p C(p-1,0)\lambda^{p-1} + \cdots + a_p C(p-1,p-1)]\end{aligned}$$

and may be written in its final form as

$$\Upsilon(\lambda) = \lambda^k(\lambda + a_1)(\lambda + a_2)....(\lambda + a_p)$$

which has k zeros and $p = n - k$ non-zero stable poles at known locations. ∎

Corollary 2 *During periods when σ_k is the outermost saturated element in the control law of Theorem 1 and the coordinate transformation used is given* *by Lemma 1, then, in steady-state, the magnitude of the k^{th} derivative, $\dot{x}_k$, is given by*

$$|\dot{x}_k| = \left|\frac{M_k}{a_{n-k}C(n-(k+1),n-(k+1))}\right| \tag{11}$$

for $k \in [1, \cdots, n-1]$

$$|\dot{x}_k| = |M_k| \tag{12}$$

for $k = n$

Proof: If σ_k is saturated, the closed loop system may be written as

$$\dot{x}_n + y_n + y_{n-1} + \cdots y_{k+1} \pm M_k = 0$$

Using Eq. (7)

$$\begin{aligned} 0 = \dot{x}_n &+ a_1 \sum_{j=0}^{0} C(0,j)x_{n-j} \\ &+ a_2 \sum_{j=0}^{1} C(1,j)x_{n-j} + \cdots \\ &+ a_{n-k} \sum_{j=0}^{n-(k+1)} C\left(n-(k+1),j\right)x_{n-j} \pm M_k \end{aligned} \tag{13}$$

When the outermost saturated element is σ_k, the dynamics eventually reach a *saturated-equilibrium* region where higher-order derivatives reach zero. So, $x^{(n)} \ldots x^{(k+1)}$, i.e., $\dot{x}_n, x_n \cdots x_{k+2}$ go to zero. The only term left from Eq. (13) is

$$a_{n-k}C(n-(k+1),n-(k+1))x_{k+1} \pm M_k = 0 \tag{14}$$

Noting that $\dot{x}_k = x_{k+1}$, rearranging Eq. (14) and taking the absolute value of both sides results in Eq. (11). Finally, when $k = n$, the outermost saturator σ_n is saturated and Eq. (13) reduces to

$$\dot{x}_n \pm M_n = 0 \tag{15}$$

Rearranging Eq. (15) and taking magnitudes of both sides results in Eq. (12) ■

Corollary 3 (Restricted Tracking) *Consider a nonlinear system with magnitude saturation at the input u given by*

$$\dot{x}_1 = x_2, \cdots, \dot{x}_n = \sigma_{n+1}(u) \tag{16}$$

and a compatible reference signal given by

$$\left[x_d(t), \quad \dot{x}_d(t), \quad \cdots \quad x_d^{(n)}(t)\right] \tag{17}$$

If $|x_d^{(n)}(t)| \leq L_{n+1} - \epsilon$ for all $t \geq t_0$ and for some $\epsilon > 0$ and given linear saturation *functions σ_i with parameters (L_i, M_i) satisfying,*

$$\begin{aligned} &L_i \leq M_i & i = 1, \cdots, n+1 \\ &M_i < \frac{1}{2}L_{i+1} & i = 1, \cdots, n-1 \\ &M_n \leq \epsilon \end{aligned}$$

then, the feedback

$$u = x_d^{(n)} - \sigma_n(y_n + \sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1)))$$

with $y = T_{yx}e$ given by Lemma 1, where, $e_i = x_i - x_d^{(i-1)}$ for $i = 1 \cdots n$, results in a globally asymptotically stable system. Additionally if conventional saturation *elements are used, the error dynamics are governed by Corollary 1 and quasi-steady rates governed by Corollary 2.*

Proof: The dynamics of Eq. (16) may be expressed in terms of the error e

$$\dot{e}_1 = e_2, \cdots, \dot{e}_n = -x_d^{(n)} + \sigma_{n+1}(u)$$

With the given control law, if the magnitude of the n^{th} derivative of the command x_d is always such that $|x_d^{(n)}(t)| \leq L_{n+1} - \epsilon$ for all $t \geq t_0$ and $M_n \leq \epsilon$, then the magnitude of the argument of σ_{n+1} is

$$|x_d^{(n)} - \sigma_n(\cdot)| \leq L_{n+1}$$

and σ_{n+1} is always in its linear region, resulting in the closed loop error dynamics becoming

$$\dot{e}_1 = e_2, \cdots, \dot{e}_n = -\sigma_n(y_n + \sigma_{n-1}(y_{n-1} + \cdots + \sigma_1(y_1))) \tag{18}$$

The conditions of this corollary and form of Eq. (18) satisfy the requirements of Theorem 1. This implies that the dynamics of e are asymptotically stable and hence x tracks x_d asymptotically. The form of Eq. (18) also allows Corollary 1 and Corollary 2 to be applied directly. ■

3 Examples

Global Stabilization: Consider the problem of stabilizing the 3^{rd} order system

$$\dot{x}_1 = x_2, \dot{x}_2 = x_3, \dot{x}_3 = u$$

using bounded control $u \in [-1, 1]$ (conventional saturation) with poles at $\{-1, -3, -2\}$. Then, $\{a_1, a_2, a_3\} = \{1, 3, 2\}$. The transformation required to achieve these poles may be expressed as

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_3(a_1a_2) & a_3(a_1+a_2) & a_3 \\ & a_2(a_1) & a_2 \\ & & a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

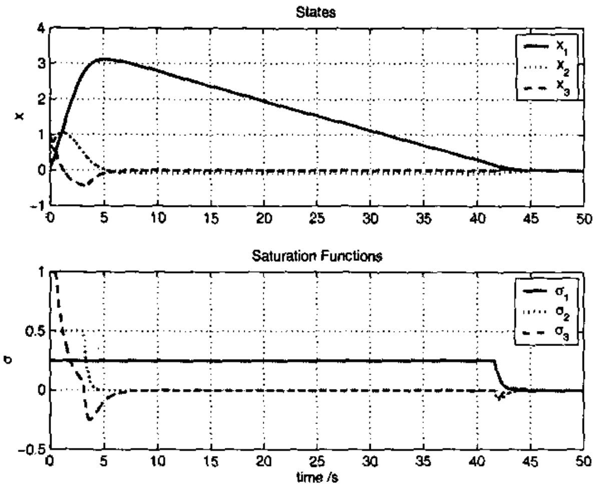

Figure 1: Initial condition response of a 3^{rd} order system

Using the nested saturation law given by Theorem 1 and choosing the saturation element parameters as follows

$$M_3 = 1, \qquad L_3 = M_3$$
$$M_2 = \frac{1}{2}L_3 - \bar{\epsilon}, \qquad L_2 = M_2$$
$$M_1 = \frac{1}{2}L_2 - \bar{\epsilon}, \qquad L_1 = M_1$$

where $\bar{\epsilon}$ is a small positive number, that is used to satisfy the inequality $M_i < \frac{1}{2}L_{i+1}$. Additionally the saturation element parameters are chosen $L_i = M_i$ (Conventional saturation). Then, the closed loop system is given by

$$\dot{x}_3 + \sigma_3(y_3 + \sigma_2(y_2 + \sigma_1(y_1))) = 0$$

An initial condition response with $x_0 = [0.1, 0.5, 1.0]$ is shown in Figure 1. The figure also shows the outputs of the saturation elements.

- 0 - 0.5s, σ_3 is saturated
- 0.5 - 3.1s, σ_2 is saturated
- 3.1 - 41.6s, σ_1 is saturated
- 41.6 - 50s, control law is unsaturated

The only region where the system practically reaches a *saturated-equilibrium* is when σ_1 is saturated, between 10 and 41 seconds. The equilibrium value for $\dot{x}_1$ is given by Corollary 2

$$|\dot{x}_1| = |x_2| = \left|\frac{M_1}{a_2 a_1}\right| = 0.0833$$

and matches the slope of x_1 in Figure 1.

Restricted Tracking: Consider a chain of 4 integrators where, $\sigma_5(u)$ represents a magnitude saturated actuator.

$$\dot{x}_1 = x_2, \dot{x}_2 = x_3, \dot{x}_3 = x_4, \dot{x}_4 = \sigma_5(u)$$

σ_5 is a conventional saturation function with parameters (L_5, M_5). A compatible command may be represented as $[x_d, \dot{x}_d, \ddot{x}_d, \dddot{x}_d, \ddddot{x}_d]$. Defining the error as, $e = x - x_d$, the error derivatives may be written as

$$\dot{e} = x_2 - \dot{x}_d$$
$$\ddot{e} = x_3 - \ddot{x}_d$$
$$\dddot{e} = x_4 - \dddot{x}_d$$
$$\ddddot{e} = \sigma_5(u) - \ddddot{x}_d$$

The control is given by Corollary 3

$$u = \ddddot{x}_d - \sigma_4(y_4 + \sigma_3(y_3 + \sigma_2(y_2 + \sigma_1(y_1))))$$

with $|\ddddot{x}_d| \leq L_5 - \epsilon$ and $M_4 \leq \epsilon$, for some $\epsilon > 0$, the saturation function parameters (L_i, M_i) chosen to satisfy the conditions given by Corollary 3 and y_i given by Lemma 1. The coordinate transformation used is $y = T_{yx}e$ where T_{yx} is given by

$$\begin{bmatrix} a_1a_2a_3a_4 & (a_1a_3 + a_2a_3 + a_1a_2)a_4 & (a_1 + a_2 + a_3)a_4 & a_4 \\ 0 & a_1a_2a_3 & (a_1 + a_2)a_3 & a_3 \\ 0 & 0 & a_1a_2 & a_2 \\ 0 & 0 & 0 & a_1 \end{bmatrix}$$

Here, the poles were taken to be at

$$\{-a_1, -a_2, -a_3, -a_4\} = \{-0.5, -1, -2, -3\}$$

The saturation function parameters were chosen as

$$L_5 = 10 \qquad M_5 = L_5 - \bar{\epsilon}$$
$$\epsilon = \frac{1}{2}L_5$$
$$L_4 = \epsilon \qquad M_4 = \epsilon - \bar{\epsilon}$$
$$L_3 = \frac{1}{2}L_4 \quad M_3 = L_3 - \bar{\epsilon}$$
$$L_2 = \frac{1}{2}L_3 \quad M_2 = L_2 - \bar{\epsilon}$$
$$L_1 = \frac{1}{2}L_2 \quad M_1 = L_1 - \bar{\epsilon}$$

where $\bar{\epsilon}$ is a small positive number chosen to satisfy $M_i < L_i$. If Corollary 2 is evaluated for various saturation elements being saturated.

$$|\dot{e}_4| = |M_4| \quad when, \sigma_4 \text{ is saturated}$$
$$|\dot{e}_3| = \left|\frac{M_3}{a_1}\right| \quad when, \sigma_3 \text{ is saturated}$$
$$|\dot{e}_2| = \left|\frac{M_2}{a_1a_2}\right| \quad when, \sigma_2 \text{ is saturated}$$
$$|\dot{e}_1| = \left|\frac{M_1}{a_1a_2a_3}\right| \quad when, \sigma_1 \text{ is saturated} \qquad (19)$$

The response of this system to a sinusoidal command compatible with $x_d = 5\sin(0.5t)$ and zero initial conditions is illustrated in Figure 2.

From Eq. (19) notice that as the bandwidth i.e., a_i is increased, the error rates in *saturated-equilibrium* decrease. Hence for higher bandwidth, the overall settling time can be higher, which is perhaps counter-intuitive. This aspect is further illustrated in Figure 3 where it is observed that the control law with faster poles (all at -1.5) takes longer to be regulated back to 0 than the system with slower poles (all at -0.5). The initial condition used was $x_0 = [0, 0.1, 1, 2]^T$ with zero command.

4 Conclusion

The extensions presented thus far provides a transformation that allows placement of poles on the non-zero real axis. Assuming the poles chosen are stable, global asymptotic stability is guaranteed. These chosen poles are guaranteed to remain constant, apart from going to zero when the respective saturation element saturates. Finally, when in *saturated-equilibrium* due to σ_k saturating, the quasi-steady rate of state change is given by a Corollary.

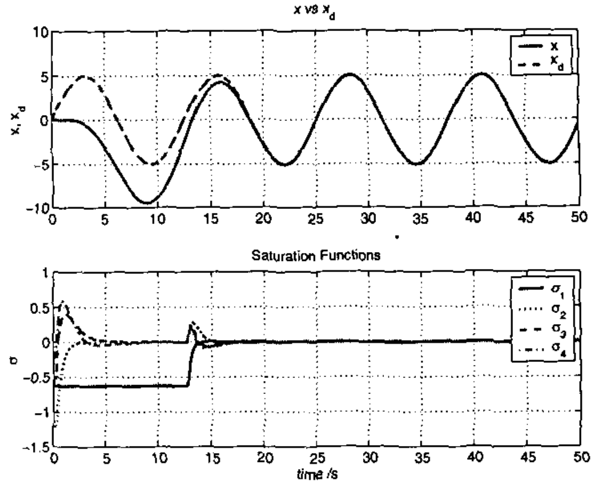

Figure 2: Response to a sinusoidal command for a 4^{th} order system

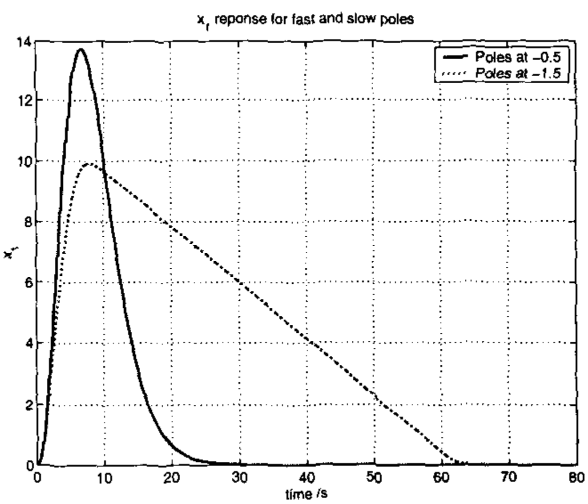

Figure 3: Comparison of the initial condition response, for a 4^{th} order system. The solid curve settles faster and has all poles at -0.5 whilst the dashed-curve settles slower and has poles at -1.5.

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